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FOURIER INTEGRAL METHODS OF PATTERN ANALYSIS

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Radiation Laboratory
Report 762-1 Jan. 21, 1946

FOURIER INTEGRAL METHODS OF PATTERN ANALYSIS

Abstract

The source of radiation $f(r)$ and its amplitude pattern $g(k)$ are a pair of Fourier Transforms, $f(r) \rightleftarrows g(k)$, where $g(k) = \int f(r)e^{ik\cdot r}dV$ and $f(r) \propto \int g(k)e^{-ik\cdot r}d\sigma$. If each point of $f_1(r)$ spreads out into a function $f_2(r)$, the resultant source is the convolution (Faltung), $f_1(r)*f_2(r)$, while its pattern is the product $g_1(k)g_2(k)$. By means of this theorem the solution of various problems can be written down by inspection. For the mattress array, $f_0*f_1*f_2*f_3 \rightleftarrows g_0g_1g_2g_3$, where f_0 is the polarization and f_1, f_2, f_3 are the linear arrays along the three edges. Other problems are: (a) Two identical antennas, (b) Binomial distribution of points, (c) Uniform distribution of points (grating), (d) Trapezoidal or triangular distribution, (e) Effect of a linear phase error. *Series Expansions:* $g(u) = \Sigma\mu_k u^k/k!$ where $u = (2\pi/\lambda)\sin\theta$ and $\mu_k = \int_{-a}^{a} x^k f(x)dx$. From this is derived the beam width W (in radians) for n db down on the main lobe. $W = A\sqrt{n}(1 - Bn)\lambda/d$ where $A = 0.152d\sqrt{\mu_0/\mu_2}$, $B = (1 - \mu_0\mu_4/3\mu_2^2)/34.7$ and $d = 2a$. For uniform illumination, $\mu_k = 2/(k+1)$ and $W = 0.529\sqrt{n}(1 - n/86.8)\lambda/d$, correct to $\pm$ 0.5% in W down to 25 db.

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A. INTRODUCTION

1. *Purpose and Scope of Report*

The development of high gain antennas for microwaves has brought into existence problems which resemble those in optics. Although the exact solution must be in agreement with electromagnetic theory, the simpler scalar theory of physical optics or sound is sufficient for many problems, particularly when the overall dimensions of the antenna contain several wavelengths. Since the radiation pattern of an antenna at a large distance can be expressed as a Fourier integral of the currents over the antenna, it is natural that certain theorems concerning the Fourier Transform are found to be useful in yielding short-cut solutions. These three dimensional theorems, when reduced to one dimension, are analogous to those of the Operational Calculus which has long been used in electrical circuit theory.

The present paper, is the first of three sections on Methods of Diffraction Pattern Analysis by the author and co-workers. Following an elementary introduction, several Fourier Integral Theorems are developed which form the basis for a number of short-cut methods. These are illustrated by numerous practical examples which include the effect of linear phasing and polarization. Section II lists tables of functions which are particularly useful in calculating the diffraction patterns of linear arrays.[1] Operational methods are extended to solve for the effect of phase errors across the aperture on the diffraction pattern. Section III deals with the scattering cross section of various shaped objects.

2. *Analogy between Microwave Antennas and Optical Reflectors*

In this section it is assumed that the reader is familiar with geometric optics and with Huygens' wave theory as presented in the usual college textbook. The representation of phase angle φ as a rotating vector, $e^{i\varphi}$, is a useful concept in studying interference and diffraction, while the concept of radiation from a dipole is essential in any electromagnetic problem.

Figures 1 to 4 illustrate a number of microwave antennas whose action is explained by elementary optics. In Fig. 1, rays which start from the focus of a paraboloidal reflector are reflected parallel to the axis, according to the laws of geometric optics. Thus, the use of paraboloidal reflectors in flash lights, search lights, reflecting telescopes, sound detectors and reflectors for microwaves, is based on the assumption that geometric optics is valid. Although geometric optics is simple to apply, its justification and its limitations depend upon the more fundamental Huygens' wave theory which is illustrated in Fig. 2. Huygens' Principle also explains the formation of a cylindrical wave from an array of point sources, (Fig. 3a) and the formation of a plane wave from a broadside array (Fig. 3b). The paraboloid and the parabolic cylinder, two types of reflectors commonly used in microwaves, are compared in Fig. 4. In Fig. 4b a linear array of

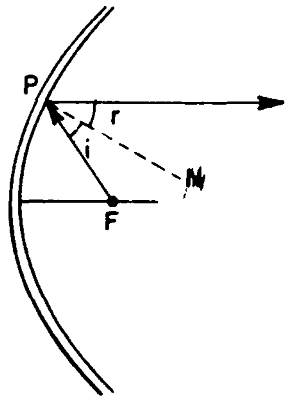

Figure 1. Paraboloidal Reflector. Law of Reflection $\angle r = \angle i$

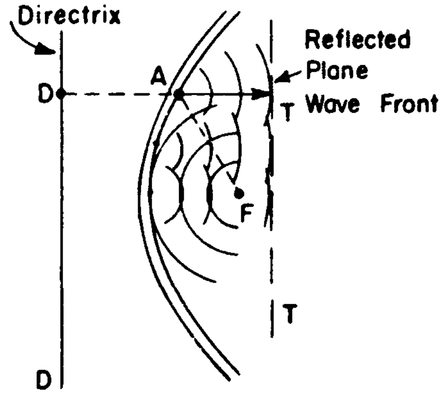

Figure 2. Paraboloidal Reflector Optical Path, FA + AT = DA + AT = constant

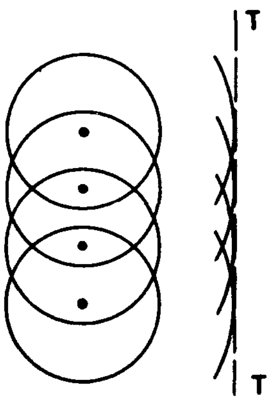

Figure 3a. Linear Array of Point Sources Form Cylindrical Wave

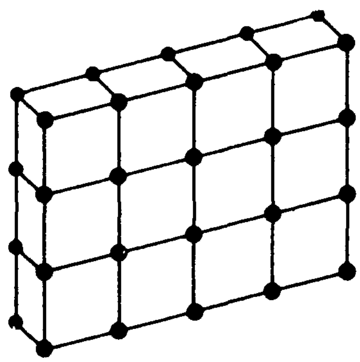

Figure 3b. Broadside Array of Point Sources Form Plane Wave

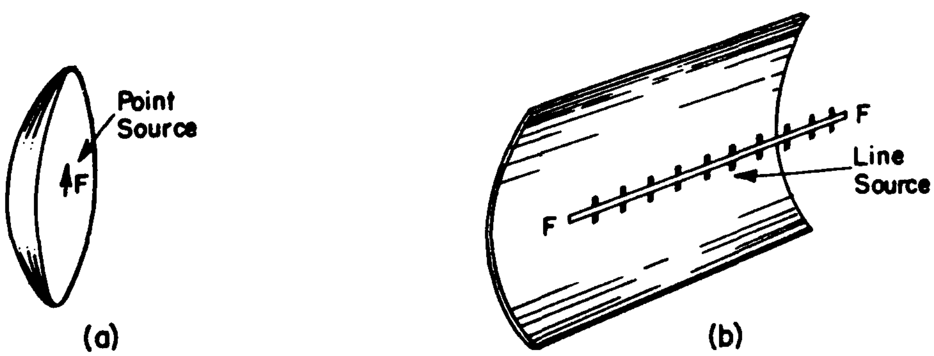

Figure 4. Comparison of (a) Paraboloid of Revolution and (b) Parabolic Cylinder.

OPTICAL EXPLANATION OF MICROWAVE ANTENNAS

HQ. AIR FORCE CAMBRIDGE RESEARCH CENTER
AIR RESEARCH AND DEVELOPMENT COMMAND
Laurence G. Hanscom Field
Bedford, Massachusetts

9 Feb 1956

SUBJECT: Radiation Laboratory Report 762-1

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RALPH E. HIATT, Acting Chief
Antenna Laboratory
Electronics Research Directorate

dipoles FF, placed at the focus of the parabolic cylinder, emits a cylindrical wave which is reflected as a plane wave.

B. POINT SOURCES

1. *Radiation from a Point Source*

The intensity of a wave in three dimensional space at a large distance R from a small source, varies according to the usual inverse square law, while the amplitude[2] varies as 1/R. An actual source may be a collection of point sources or an equivalent continuous distribution of point sources placed at a reflector.

In the case of longitudinal waves such as sound, the typical point source is a tiny sphere whose radius pulsates so as to emit a spherical wave. Surfaces of constant amplitude are spheres concentric with the source.

In the case of electromagnetic waves which are transverse waves, the electric field E, the magnetic field H and the direction of propagation are perpendicular to each other. The typical point source is a dipole which may be considered to be a very short wire of length L carrying a sinusoidal current of I amperes. At distances large compared to $\lambda/2\pi$, where λ is the wavelength, the electric field in volts per meter is given by:[3]

$$E = 60\pi \frac{IL}{\lambda} \frac{\sin \psi}{R} \tag{1}$$

ψ is the angle between the line of sight and the dipole, and R is the range in meters. The ratio L/λ is assumed small. When a collection of dipoles have parallel axes it is possible to treat them first as point sources, inserting the polarization factor $\sin \psi$ at the end.

The motion of a vibrating sound source, or the current in a radiating dipole, may be represented by one coordinate of a point which moves in a circle with a uniform angular velocity, $\omega = 2\pi f$. This is represented by a vector which has turned through a phase angle $\varphi = \omega t$, which is equivalent to multiplication by $e^{i\omega t}$.

2. *Radiation from a Collection of Point Sources*

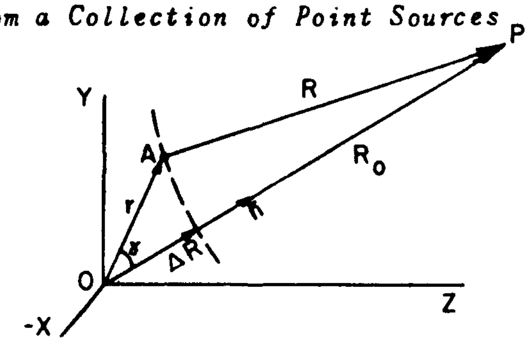


Fig. 5 Source point at A. Field point at P. The range relative to the origin is $\Delta R = \bar{r} \cdot \bar{n}$.

In Fig. 5 the typical point source A is a distance r from the origin and a distance R from the field point P which in turn is a distance R_o from the origin. Since the radiation travels from A to P in the time R/c, the phase at P is retarded by an amount $\omega R/c$. The field at P due to a number of such point sources will be, according to Huygens' Principle, the sum of the individual fields[4,5].

$$E \propto \sum \frac{A_n}{R} e^{i\omega(t-R/c)} \tag{2}$$

If we let $R = R_o - \Delta R$ and take out the factor common to all terms, (2) becomes

$$E \propto \frac{e^{i\omega(t-R_o/c)}}{R_o} \sum \frac{A_n \, e^{i\omega\Delta R/c}}{1-\Delta R/R_o} \tag{3}$$

The external factor represents a spherical wave traveling outward with a velocity of light, since the phase is constant if $R_o = ct$, and the amplitude dimishes as $1/R_o$. A small section of the spherical wave may be considered part of a plane wave with a normal $\bar{n}$ in the direction OP. Henceforth we shall omit the external factor and consider only the contribution within the summation sign, which is proportional to the amplitude of the resultant plane wave. As R_o becomes sufficiently large the ratio $\Delta R/R_o$ in the denominator becomes negligibly small, but the ΔR in the exponent must be retained as the phase φ relative to the origin is not small.

$$\varphi = \frac{\omega \, \Delta R}{c} = \frac{2\pi f \Delta R}{c} = 2\pi \frac{\Delta R}{\lambda} \tag{4}$$

An approximate value of ΔR is the component of $\bar{r}$ in the direction of $\bar{n}$ as shown in Fig. 5

$$\Delta R = r \cos \gamma = \bar{r} \cdot \bar{n} \tag{5}$$

($\bar{r} \cdot \bar{n}$ is the scalar product of the two vectors $\bar{r}$ and $\bar{n}$, while γ is the angle between them). If the phase change per unit length in the direction of $\bar{n}$ is designated by the vector propagation constant $\bar{k}$, then (4) reduces to

$$\varphi = \frac{2\pi \bar{n} \cdot \bar{r}}{\lambda} = \bar{k} \cdot \bar{r} \tag{6}$$

and the field (3) is proportional to a function of $\bar{k}$.

$$g(k) = \sum A_n e^{i\bar{k} \cdot \bar{r}} \tag{7}$$

Thus, the amplitude of the equivalent plane wave at a unit distance from the origin is g(k), which is the sum of the individual phase vectors, weighted according to the strengths of the source points.

3. *Examples of Diffraction Patterns*Derived from Point Sources*

We shall now consider the examples of isotropic point sources illustrated in Fig. 6. By definition, a unit (isotropic) point source placed at the origin (Case 1), radiates a unit amplitude at a unit distance regardless of direction. Hence, $g(k) = 1$. When the point source is displaced a distance $\bar{r}_1$ from the origin as in Case 2, the phase angle is advanced by an amount $\varphi = \bar{k}\cdot\bar{r}_1 = (2\pi r_1/\lambda)\cos\gamma$. Note that φ has the same value for all lines of sight that make an angle γ with $\bar{r}_1$. The locus of these directions forms the cone illustrated in Case 2.

When two equal point sources of equal sign (Case 3), or of opposite sign (Case 4), are placed at $\pm\bar{r}_1$, the diffraction patterns are respectively the sum and difference of two exponential terms.

$$\text{Case 3.} \qquad g(k) = e^{i\bar{k}\cdot\bar{r}_1} + e^{-i\bar{k}\cdot\bar{r}_1} = 2\cos\bar{k}\cdot\bar{r}_1 = 2\cos\varphi \qquad (8)$$

$$\text{Case 4.} \qquad g(k) = e^{i\bar{k}\cdot\bar{r}_1} - e^{-i\bar{k}\cdot\bar{r}_1} = 2i\sin\bar{k}\cdot\bar{r}_1 = 2i\sin\varphi \qquad (9)$$

One must remember that for any arrangement of point sources along a line, the pattern is identical over the cone of directions shown in Case 2.

The diffraction pattern for the uniform line source (Case 5), extending from $-\bar{r}_1$ to $\bar{r}_1$, may be obtained by integrating the patterns of a uniform density of point sources of the type of Case 2.

$$g(\bar{k}) = \int_{-\bar{r}_1}^{\bar{r}_1} e^{i\bar{k}\cdot\bar{r}}\,dr = \int_{-\varphi}^{\varphi} \frac{e^{i\varphi}\,d\varphi}{|k|\cos\gamma}$$

$$= \frac{2\sin\varphi}{|k|\cos\gamma} = 2r_1\frac{\sin\varphi}{\varphi} \qquad (10)$$

This is the diffraction pattern of a single slit commonly derived in optics. Other examples of point sources will be taken up after the discussion of the properties of Fourier Transforms.

C. THE SOURCE AND ITS PATTERN AS A PAIR OF FOURIER TRANSFORMS

1. *General*

We will now proceed to show that the source function and its diffraction pattern are Fourier Transforms. Let us designate the source function by $f(\bar{r})$. This may either be the set of source points $[A_n r_n]$ described above or a continuous function $f(\bar{r})$ of the position vector $\bar{r}$. The amplitude diffraction pattern (7) now takes the form of a Fourier Integral[6,7]

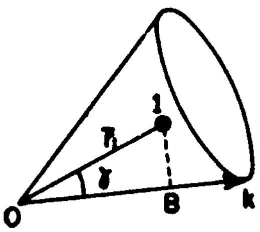
**The term diffraction pattern as employed in optics is here used interchangeably with the term radiation pattern which is employed in radio.*

$f(\bar{r}) = [A_n \bar{r}_n]$ | $g(\bar{k}) = \sum A_n \, e^{i\bar{k}\cdot\bar{r}}$

Case 1. Unit Point Source at the origin

$g(\bar{k}) = 1$

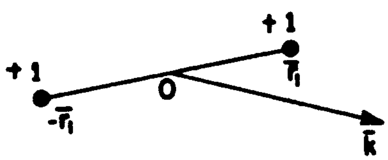
Case 2. Unit Point Source at $\bar{r} = \bar{r}_1$

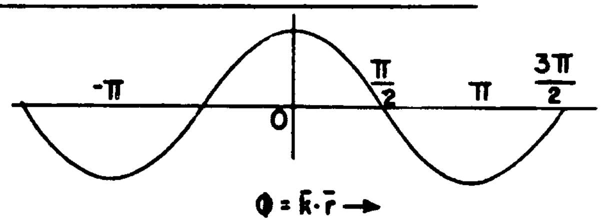


$g(\bar{k}) = e^{i\bar{k}\cdot\bar{r}} = e^{i\varphi}$

$\varphi = kr_1 \cos\gamma = (2\pi r_1/\lambda)\cos\gamma$

Case 3. Two Point Sources, +1 at $\bar{r}_1$, +1 at $-\bar{r}_1$

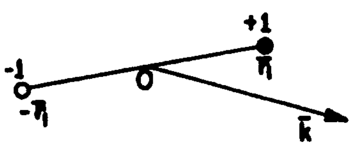


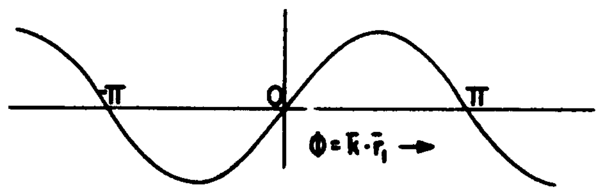


$g(k) = e^{i\bar{k}\cdot\bar{r}_1} + e^{-i\bar{k}\cdot\bar{r}_1} = 2\cos\varphi$

Zeros for $\varphi = \pi/2,\ 3\pi/2$, etc.

Case 4. Two Point Sources, +1 at $\bar{r}_1$, -1 at $-\bar{r}_1$

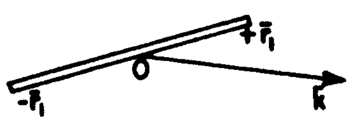


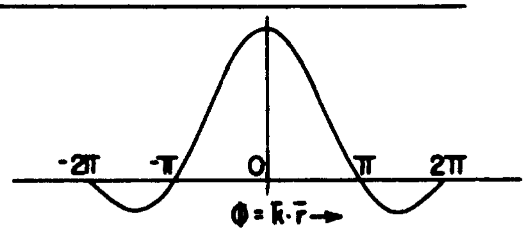


$g(\bar{k}) = e^{i\bar{k}\cdot\bar{r}_1} - e^{i\bar{k}\cdot\bar{r}_1} = 2i\sin\varphi$

Zeros for $\varphi = 0,\ \pi,\ 2\pi$, etc.

Case 5. Uniform Line Source from $-\bar{r}_1$ to $+\bar{r}_1$





$g(\bar{k}) = \int_{-\bar{r}_1}^{\bar{r}_1} e^{i\bar{k}\cdot\bar{r}} dr = 2\bar{r}_1 \dfrac{\sin\varphi}{\varphi}$

Zeros for $\varphi = \pi,\ 2\pi,\ 3\pi$, etc.

FIGURE 6. DIFFRACTION PATTERNS OF ISOTROPIC POINT SOURCES

$$g(\bar{k}) = \int f(\bar{r})\, e^{i\bar{k}\cdot\bar{r}} dV \qquad (11)$$

The integration is carried out over the *volume* occupied by the source, the element of volume being dV.

As previously mentioned, $g(\bar{k})$ may be considered to be the amplitude of a plane wave emitted by the source in the direction of $\bar{n}$ or $\bar{k}$. If we reverse these waves they will, by the principle of reversibility in optics, retrace their paths exactly, converging to the source function $f(\bar{r})$. Thus, we may represent $f(\bar{r})$ as the sum of *plane waves*.

$$f(\bar{r}) = \int g(\bar{k})\, e^{-i\bar{k}\cdot\bar{r}} d\sigma \qquad (12)$$

The integration is carried out over the *surface* of a unit sphere, the element of solid angle being dσ. Note that the $\bar{k}$ in the phase factor is reversed because of the reversal of direction. We have here ignored a normalization factor on the right side of (12).

Equations (11) and (12) are symmetric in their mathematical form and are known as a *pair of Fourier Transforms*. This fact is sometimes denoted by the double arrow symbol

$$f(\bar{r}) \rightleftarrows g(\bar{k}) \qquad (13)$$

Again, the statement that " $g(\bar{k})$ is the Fourier Transform of $f(\bar{r})$ " may be denoted by

$$g(\bar{k}) = T\, f(\bar{r}) \qquad (14)$$

2. *Fourier Transforms in Various Coordinate Systems*

When the amplitude function is known over a simple geometric surface such as a sphere or a cylinder, it can be expanded in a series of orthogonal functions, each of which has its own Fourier Transform. These are given by Stratton[8].

The case of rectangular coordinates is quite simple. Let the amplitudes of the i, j and k components of the vector $\bar{k}$ be u, v and w. If x, y and z are the corresponding components of $\bar{r}$, the phase becomes

$$\varphi = \bar{k}\cdot\bar{r} = (ui+vj+wk)\cdot(xi+yj+zk) = (ux+vy+wz) \qquad (15)$$

The volume element dV is dxdydz while the volume element in diffraction space, assuming a *variation* in *wavelength*, becomes $dudvdw/(2\pi)^3$. If we divide the numerical factor $(2\pi)^{-n}$ into two symmetrical factors $(2\pi)^{-n/2}$ we arrive at the usual *three dimensional* transforms[8]

$$g(u,v,w) = \frac{1}{(2\pi)^{3/2}} \iiint f(x,y,z)\, e^{i(ux+vy+wz)} dxdydz \qquad (16)$$

$$f(x,y,z) = \frac{1}{(2\pi)^{3/2}} \iiint g(u,v,w)\, e^{-i(ux+vy+wz)}\, du\,dv\,dw \qquad (17)$$

In case $f(x,y,z)$ has zero thickness in the i,j or k direction $g(u,v,w)$ will be independent of the corresponding u,v or w variable due to the fact that this variable vanishes from the expression for the phase, (15). Thus, in the case of two dimensions, which is particularly useful for broadside arrays

$$g(u,v) = \frac{1}{2\pi} \iint f(x,y)\, e^{i(ux+vy)}\, dx\,dy \qquad (18)$$

$$f(x,y) = \frac{1}{2\pi} \iint g(u,v)\, e^{-i(ux+vy)}\, du\,dv. \qquad (19)$$

In the case of one dimension, used in the analysis of linear arrays

$$g(u) = \frac{1}{\sqrt{2\pi}} \int f(x)\, e^{iux} dx \qquad (20)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int g(u)\, e^{-iux} du\,. \qquad (21)$$

Usually in antenna problems we wish to keep the frequency constant which imposes the condition that

$$u^2 + v^2 + w^2 = k_o^2 \qquad (22)$$

This condition restricts the diffraction space to the surface of the sphere mentioned in connection with the inverse transform (12).

Again, we may wish to keep the direction fixed and merely vary the frequency. In case the direction is along the x-axis, the one dimensional problem reduces to the frequency spectrum $g(\omega)$ of a transient $f(t)$ as treated in electrical circuit theory[9]. There is a change of scale arising from the relationships $\varphi = ux = \omega t$, $x = ct$.

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int f(t)\, e^{i\omega t} dt \qquad (23)$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int g(\omega)\, e^{-i\omega t} d\omega \qquad (24)$$

The normalization constants for any of the above equations are chosen so that there is conservation of energy or conservation of power in both the source space and diffraction space.

$$\int |f|^2 dV \quad = \quad \int |g|^2 d\sigma \tag{25}$$

3. *Effect of Scaling*

If the amplitude of a function f(x) is multiplied by the factor s while it is made narrower in the same ratio so as to preserve its area, its Fourier Transform will be simply broadened by the factor s, while its peak is unchanged. In mathematical notation, let

$$f(x) \rightleftarrows g(u) \tag{26}$$

then $$sf(sx) \rightleftarrows g(u/s) \tag{27}$$

To prove this, let sx = x' and u/s = u', then ux = u'x'. The Fourier Transform of sf(sx) is

$$\frac{1}{\sqrt{2\pi}} \int sf(sx)e^{iux}dx = \frac{1}{\sqrt{2\pi}} \int f(x')e^{iu'x'}dx' = g(u')$$

which proves (27). This principle may be applied to any of the three dimensions independently.

4. *Fourier Transforms of Even and Odd Functions*

Since $e^{iux} = \cos ux + i \sin ux$, g(u) can be written in the form

$$g(u) = \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{a} f(x) \cos ux \, dx + i \int_{-a}^{a} f(x) \sin ux \, dx \right]$$

$$= g_e(u) + ig_o(u). \tag{28}$$

The first integral is zero for an odd function and therefore applies only to an even function, f_e, while the second applies only to an odd function, f_o. The presence of the i indicates that the two transforms are 90^o out of phase. This was discussed in more detail[10] in R.L. Report T-7. It follows that the power pattern is the sum of squares of the individual patterns.

$$P(u) = g(u)\,\bar{g}(u) = (g_e + ig_o)\,(g_e - ig_o) = g_e^2 + g_o^2 \tag{29}$$

This is a real positive function with no zeros since g_e and g_o are unlikely to have coincident zeros.

It will be the policy in the remainder of this paper to use particular coordinate systems only when they simplify the problem.

5. *Fourier Transform of the Complex Conjugate*

The power $P(k)$ in the diffraction pattern is of practical importance in antenna theory and measurements. This is the square of the absolute magnitude of $g(\bar{k})$ or the product of its complex conjugates.

$$P(\bar{k}) = |g(\bar{k})|^2 = g(\bar{k})\hat{g}(\bar{k}) \tag{30}$$

Now, to find the complex conjugate $\hat{g}(\bar{k})$ we simply replace the quantities in (11) by their complex conjugates which includes reversing the i in the exponent. Thus,

$$\hat{g}(\bar{k}) = \int \hat{f}(\bar{r}) e^{-i\bar{k}\cdot\bar{r}} dV \tag{31}$$

We may make the exponent positive again by reversing all k's or reversing all r's. These changes lead respectively to the transforms

$$\hat{f}(\bar{r}) \rightleftharpoons \hat{g}(-\bar{k})$$

$$\hat{f}(-\bar{r}) \rightleftharpoons \hat{g}(\bar{k}) \tag{32}$$

6. *Effect of a Linear Phase Change*

Let $g_1(\bar{k})$ be the diffraction pattern when the source function $f_1(\bar{r})$ is real, i.e. all source points are vibrating in phase. The effect, then, of exciting them by means of a plane wave of wavelength λ_o traveling in the direction $\bar{n}_o$, is to introduce a linear phase lag φ_o at each point (See Fig. 7).

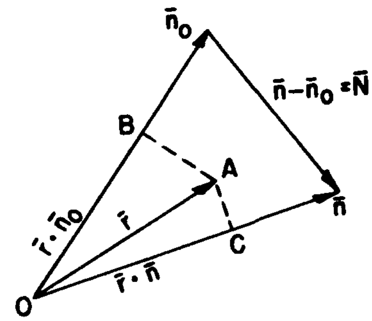


Fig. 7. Path Difference when Excited by Plane Wave

$$OC-OB = \bar{r}\cdot\bar{n}-\bar{r}\cdot\bar{n}_o = \bar{r}\cdot(\bar{n}-\bar{n}_o) = \bar{r}\cdot\bar{N}$$

$$\varphi_o = 2\pi\frac{\overline{OB}}{\lambda_o} = \frac{2\pi}{\lambda_o}\ \bar{r}\cdot\bar{n}_o = \bar{r}\cdot\bar{k}_o \tag{33}$$

We then have a complex source function

$$f(\bar{r}) = e^{-i\bar{r}\cdot\bar{k}_o} f_1(\bar{r}) \tag{34}$$

ich results in the diffraction pattern

$$g(\bar{k}) = \int f(\bar{r})\, e^{i\bar{r}\cdot\bar{k}}\, dV = \int f_1(\bar{r})\, e^{i\bar{r}\cdot(\bar{k}-\bar{k}_o)}\, dV = g_1(\bar{k}-\bar{k}_o) \qquad (35)$$

By adding $\bar{k}_o$ to each abscissa, we obtain

$$g(\bar{k}+\bar{k}_o) = g_1(\bar{k}). \qquad (36)$$

Thus, the effect of a linear phase error is to shift each point in the diffraction pattern $g_1(\bar{k})$ from the direction associated with $\bar{k}$ to that of $(\bar{k}+\bar{k}_o)$. This theorem for three dimensional space corresponds to the Heaviside shift theorem in the operational calculus of electric circuits.

In the case of a linear array excited by a wave traveling along the wave guide, the phase φ is shifted by an amount $\varphi_o = 2\pi r_1/\lambda_g$, where λ_g is the wave guide wavelength. Thus, the resultant diffraction pattern is a new function of φ,

$$g(\varphi) = g_1(\varphi-\varphi_o) \qquad (37)$$

where $g_1(\varphi)$ is the pattern for an array in phase. (We are here using $g(\varphi)$ and $g(\bar{k})$ interchangeably although they really are different due to the scale factor between φ and $\bar{k}$). Since $g_1(\varphi)$ has a maximum when φ is zero, $g(\varphi-\varphi_o)$ will have a maximum when $(\varphi-\varphi_o) = 0$, or when

$$\cos\gamma = \frac{\lambda}{\lambda_g} = \frac{V_g}{V} \qquad (38)$$

V_g is the velocity in the guide. This is illustrated in Fig. 8a where a wavelet travels a distance OA in the guide while it is traveling a distance OB outside. The line AB is a common wavefront for all wavelets along the guide. The same construction is used to explain the critical angle at the boundary of a dielectric, bow waves from boats on the water, or the shock waves from bullets the air, Fig. 8c.

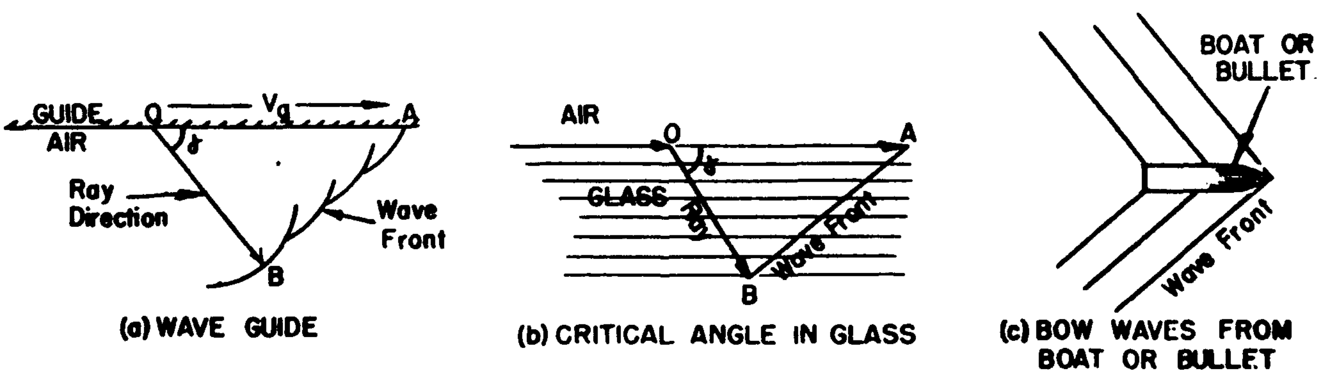


Fig. 8. Huygens' Wave Construction for Linear Phase Errors

If $\lambda = \lambda_g$, then the peak intensity occurs when γ is zero and the result is an end-fire array. An interesting example of an end fire is the case of two point sources with a spacing, $d = \lambda/4$ and with a phase difference of $\pi/2$, or $\pm \pi/4$ from that of the origin. The solution is to shift the curve $g_1(\varphi) = \cos\varphi$ to the right an amount $\varphi_0 = \pi/4$, so that the peak value occurs at $\gamma = 0$, and the first and only zero at $\gamma = 180^\circ$.

D. THE CONVOLUTION THEOREM

1. *Derivation of $f_1 * f_2 \rightleftarrows g_1 g_2$*

The convolution theorem, known in mathematics as the Parseval theorem, is a powerful tool for solving many diffraction problems. The following derivation for *three dimensional space* holds equally well for one[11] or two dimensions. Let us consider the two source functions $f_1(\bar{r})$ and $f_2(\bar{s})$ of Fig. 9.

$$f_1(\bar{r}) = [a_n \bar{r}_n] \;, \qquad f_2(\bar{s}) = [b_m \bar{s}_m] \tag{39}$$

Thus, $f_1(\bar{r})$ is a "set" of source points with weights a_n and position vectors $\bar{r}_n$ in three dimensional space, while $f_2(\bar{s})$ has weights b_m and position vectors $\bar{s}_m$. Note that the square bracket is employed for a "set" instead of the Σ which would have denoted a sum.

The convolution of f_1 and f_2, also termed the resultant or the "Faltung" (folding), consists of spreading each point of one of the functions out into the other function. This is illustrated in Fig. 10a. Thus, the point $a_n \bar{r}_n$ is spread into the series $a_n[b_m(\bar{r}_n + \bar{s}_m)]$ while the convolution f is given by

$$f = f_1 * f_2 = \left[a_n \left[b_m(\bar{r}_n + \bar{s}_m)\right]\right] \tag{40}$$

The star between two functions indicates their convolution. This is symmetrical with respect to f_1 and f_2, the typical term having a weight $a_n b_m$ and a position vector $(\bar{r}_n + \bar{s}_m)$. (See Fig. 10b). Substitution of these values in (7) yields the diffraction pattern

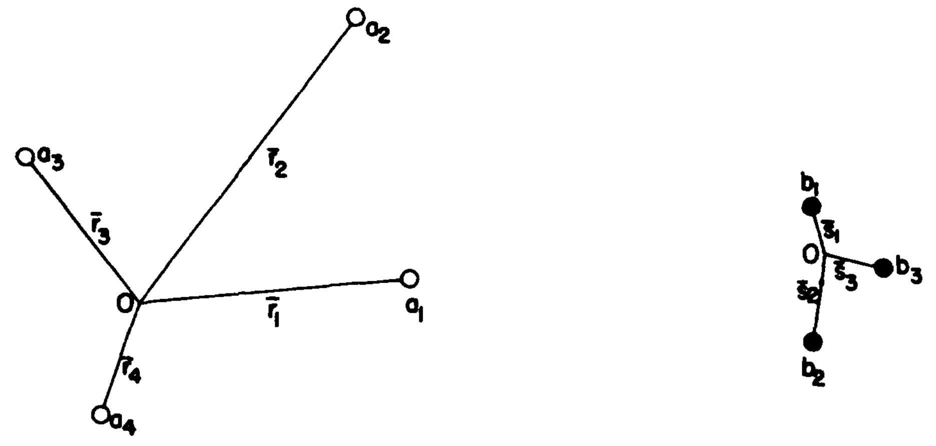
$$g = \sum a_n \sum b_m e^{ik\cdot(\bar{r}_n + \bar{s}_m)} \tag{41}$$

But this is identical with the product of the transforms of f_1 and f_2,

$$g_1 g_2 = \left(\sum a_n e^{i\bar{k}\cdot\bar{r}_n}\right)\left(\sum b_m e^{i\bar{k}\cdot\bar{s}_m}\right) \tag{42}$$

Since in either case the typical term is $a_n b_m e^{i\bar{k}\cdot(\bar{r}_n + \bar{s}_m)}$. Thus we arrive at the convolution theorem which states that *the transform of the convolution is the product of the transforms.*

$$f_1 * f_2 \rightleftarrows g_1 g_2 \tag{43}$$



(a) $f_1(\bar{r}) = [a_n \bar{r}_n]$ (b) $f_2(\bar{s}) = [b_m \bar{s}_m]$

Figure 9. Point functions $f_1(\bar{r})$ and $f_2(\bar{s})$, where $\bar{r}$ and $\bar{s}$ are position vectors in three dimensional space.

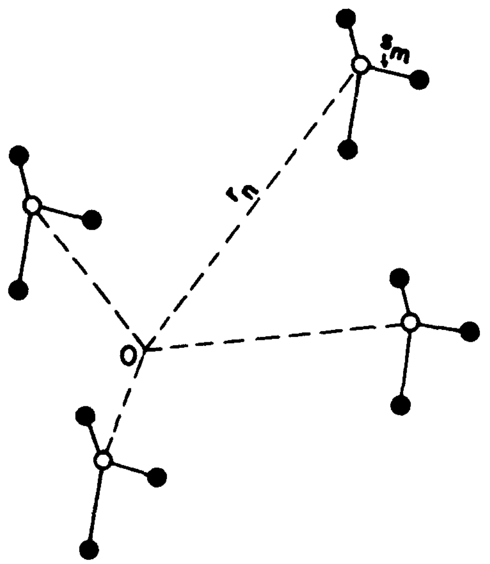


Figure 10a. Convolution of f_1 and f_2
$f = f_1 * f_2$

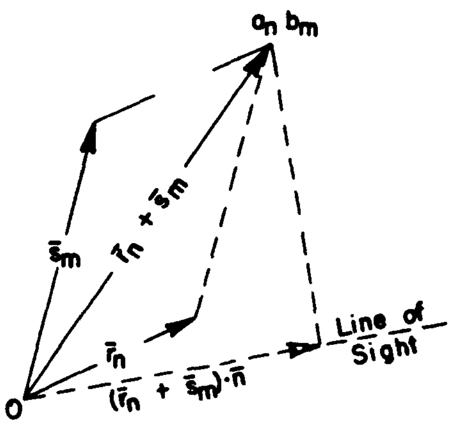


Figure 10b. Typical term in convolution. Weighted points $a_n b_m$ at position $\bar{r}_n + \bar{s}_m$.

FIGURES 9, 10. CONVOLUTION OF POINT SOURCES IN SPACE

This theorem holds equally well for continuous functions. It is quite apparent that we may form the convolution of a third function f_3 with f, which is already the convolution of f_1 and f_2. Again, the order of operations is immaterial.

$$f_1 * f_2 * f_3 * \dots f_{n-1} * f_n \rightleftarrows g_1 g_2 g_3 \cdot \cdot \cdot g_n \qquad (44)$$

Due to the symmetry between the f and g functions, the above convolution properties hold if we interchange the f's and g's.

$$f_1 f_2 \rightleftarrows g_1 * g_2 \qquad (45)$$

The convolution theorem is so simple that frequently by the time one has clearly stated the problem the solution can be written down. Numerous illustrations of the method are given later in the paper.

2. *Integral Representation of the Convolution*

We shall now derive a mathematical expression for the convolution. Let us change notation, replacing $\bar{r}$ by $\bar{r}_1$, $\bar{s}$ by $\bar{r}_2$, letting $\bar{r}$ stand for the vector sum $(\bar{r}_1+\bar{r}_2)$. We now allow the volume element $f_1(\bar{r}_1)dV_1$, considered as a mass point, to spread out into the function $f_2(\bar{r}_2)$. The convolution $f(\bar{r})$ is the integral over all such elements of volume.

$$\begin{aligned} f(\bar{r}) = f_1(\bar{r}_1) * f_2(\bar{r}_2) &= \int f_1(\bar{r}_1) f_2(\bar{r}_2)\, dV_1 \\ &= \int f_1(\bar{r}_1) f_2(\bar{r}-\bar{r}_1)\, dV_1 \end{aligned} \qquad (46)$$

If the functions were constrained to one dimension we would have the usual expression for the convolution of $f_1(x)$ and $f_2(x)$

$$f(x) = f_1(x) * f_2(x) = \int f_1(\alpha) f_2(x-\alpha)\, d\alpha \qquad (47)$$

where α is a variable of integration.

3. *The Parseval Formulas. Conservation of Energy*

Writing (46) as the transform of $g_1 g_2$

$$\int f_1(\bar{r}_1) f_2(\bar{r}-\bar{r}_1)\, dV_1 = \int g_1(\bar{k}) g_2(\bar{k}) e^{-i\bar{k}\cdot\bar{r}} d\sigma \qquad (48)$$

If $|r| = 0$

$$\int f_1(\bar{r}_1) f_2(-\bar{r}_1)\, dV_1 = \int g_1(\bar{k}) g_2(\bar{k})\, d\sigma$$

Now let $f_2(-\bar{r}_1) = \hat{f}_1(-\bar{r}_1)$, whereupon $g_2(\bar{k})$ becomes $\hat{g}_1(\bar{k})$ according to (32).

Dropping the subscript 1,

$$\int f(\bar{r})\hat{f}(\bar{r})\,d\rho = \int g(\bar{k})\hat{g}(\bar{k})\,d\sigma$$

or

$$\int |f(\bar{r})|^2\,d\rho = \int |g(\bar{k})|^2\,d\sigma \tag{49}$$

The above Parseval formulas are usually developed in the literature for one dimension only. Eq. (49) is really a statement of the conservation of energy between the source space and diffraction space, or between antenna primary patterns and secondary patterns.

4. *The Power Diffraction Pattern as the Transform of* $f(\bar{r}) * \hat{f}(-\bar{r})$

In most measurements and experiments involving diffraction patterns, the phase of the pattern is not detected, only the power $P(\bar{k}) = g(\bar{k})\hat{g}(\bar{k})$ being significant. But according to (30), (32) and (43), $P(\bar{k})$ is the transform of the convolution of $f(\bar{r})$ and $\hat{f}(-\bar{r})$ showing that this is the significant characteristic of the antenna

$$f(\bar{r}) * \hat{f}(-\bar{r}) \rightleftharpoons g(\bar{k})\hat{g}(\bar{k}) = P(\bar{k}) \tag{50}$$

The same equation is obtained from the Parseval Formulas (46, 48) by substituting $g(\bar{k})$ for $g_1(\bar{k})$ and $\hat{g}(k)$ for $g_2(\bar{k})$.

E. EXAMPLES OF CONVOLUTIONS IN SPACE

1. *Effect of a Displacement*

The effect of a displacement of the origin of a source function $f_1(\bar{r})$ to a position $\bar{r}_o$ can be considered to be the convolution of $\bar{r}_o$ and $f_1(\bar{r})$

$$f(\bar{r}) = f_1(\bar{r}-\bar{r}_o) = \bar{r}_o * f_1(\bar{r}) \tag{51}$$

The resultant diffraction pattern is then the product of the two patterns

$$g(\bar{k}) = e^{i\bar{k}\cdot\bar{r}_o}\, g_1(\bar{k}) \tag{52}$$

Note that this introduces a *linear phase error* into the diffraction pattern.

2. *Effect of a Linear Phase Error*

This effect, discussed earlier, is the same as the above example with the roles of f and g, and those of $\bar{k}$ and $\bar{r}$ reversed. The resultant source function is the product

$$f(\bar{r}) = e^{-i\bar{r}\cdot\bar{k}_o}\, f_1(\bar{r})$$

while the resultant diffraction pattern is that derived earlier by a direct method. (See (35).)

$$g(\bar{k}) = g_1(\bar{k}-\bar{k}_o) \tag{52a}$$

3. *Effect of Polarization*

If the currents in an antenna are all parallel we may consider that each scalar point has been spread out into a dipole. The resultant source function is the convolution of the scalar source function $f(\bar{r})$ and the dipole, so that the diffraction pattern is the product of the scalar diffraction pattern and the factor $\sin\psi$ of (1).

4. *Broadside Mattress Type Array*

The mattress type array of Fig. 3b consists of two rectangular layers of equally spaced dipoles with their axes parallel and with corresponding dipoles in the front and back layers arranged for end-fire. If all dipoles are equally excited, the mattress is the convolution of four functions, the polarization f_0 and the three linear arrays along the three edges. The diffraction pattern is therefore the product of the four patterns according to (44).

$$f_0 * f_1 * f_2 * f_3 \rightleftharpoons g_0 g_1 g_2 g_3 \qquad (53)$$

This equation applies to a much wider class of antennas than the conventional mattress type, since the individual axes may have any orientation, and the spacing, amplitude and phase along any one linear array may be arbitrary. The case of a three dimensional grating with inclined axes is treated by Stratton.[12]

5. *Two Identical Antennas*

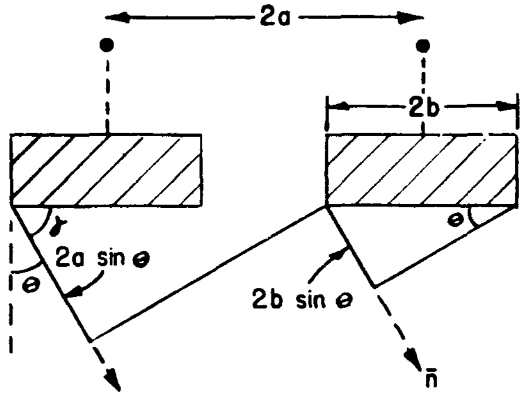
The arrangement of two identical antennas placed with centers at $\pm\bar{r}_2$ is simply the convolution of one of the antennas with two points located at $\pm\bar{r}_2$. The diffraction pattern is then the product of the pattern $g_1(\bar{k})$ for one of the antennas and the interference pattern for two point sources, either in phase, as in Case 4., (Fig. 6), or with an arbitrary phase difference φ_0 due to the length of feed line. If we also include the polarization factor we can write the complete pattern as the product of three factors

$$g(\bar{k}) = g_1(\bar{k}) \cos(\bar{k}\cdot\bar{r}_2 - \varphi_0) \sin\psi \qquad (54)$$

The next paragraph discusses a special case.

6. *Double Slit Source Considered as a Convolution*

The double slit source of optics, illustrated in Fig. 11, can be considered to be the convolution of a pair of points situated at $\pm\bar{a}$ and a single slit of width 2b. The over-all diffraction pattern is the product of the two patterns $2\cos au$ and $2(\sin bu)/u$ illustrated in Fig. 6. It is recalled that the first factor is unity when au is a multiple of π or when the path difference $(2a\sin\theta) = (2a\cos\gamma)$ between centers is a whole number of wavelengths, so that the two sources are in phase. The second is zero when bu is a multiple of π or when the variation in path length $(2b\sin\theta)$ over a single slit is a whole number of wavelengths.



Diffraction Patterns For:

Two Points, $g_1(u) = 2 \cos au$

One Slit, $g_2(u) = 2 \dfrac{\sin bu}{u}$

Two Slits $g(u) = g_1 \, g_2$

$= 4 \cos au \dfrac{\sin bu}{u}$

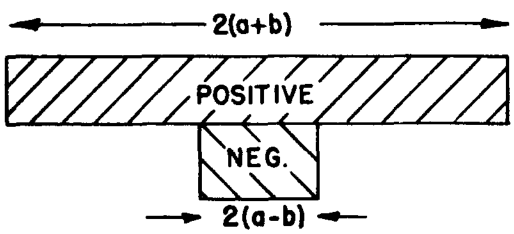
Fig. 11. Double Slit as the Convolution of Two Points and a Slit

7. Double Slit as the Difference of Two Slits

An alternate method of attack is to make use of the fact that Fourier Transforms are linear transformations, i.e. the transform of a sum of functions is the sum of their transforms.

$$f_1(x) + f_2(x) + \ldots \rightleftharpoons g_1(u) + \ldots \; g_2(u) + \ldots \tag{55}$$

Fig. 12 shows how the difference of two slits of widths 2(a+b) and 2(a-b) is equivalent to the double slit of Fig. 11. The corresponding diffraction patterns in the two figures are also equivalent since they are trigonometric identities.



$$g(u) = 2 \frac{\sin(a+b)u}{u} - 2 \frac{\sin(a-b)u}{u}$$

$$= 4 \cos au \frac{\sin bu}{u}$$

Fig. 12. Double Slit as the Difference of Two Slits of Widths 2(a+b) and 2(a-b)

F. CONVOLUTIONS OF PAIRS OF POINTS IN ONE DIMENSION

1. *Successive Convolutions of Pairs of Points*

Fig. 13 illustrates successive convolutions of pairs of points in one dimension. The top row shows one pair at $\pm\bar{a}$ with a diffraction pattern cos au. The next row shows its convolution with a second pair of points at $\pm\bar{b}$, the resulting pattern being cos au cos bu. The third row is the convolution of the second row with a pair of points at $\pm\bar{c}$, the pattern being cos au cos bu cos cu. This process may be continued indefinitely.

The final pattern is zero whenever one of the component patterns is zero. If $\bar{a}$, $\bar{b}$ and $\bar{c}$ differ slightly, their first zeros differ slightly so that the pattern is almost zero over a range of angles. The pattern can become unity at a point other than the origin only if $\bar{a}$, $\bar{b}$ and $\bar{c}$ have simple integral ratios. Thus, the presence of side lobes due to equidistant supports can be reduced only by a careful study of variable spacing.

2. *Binomial Distribution of Points*

Fig. 14 shows the binomial distribution of n points resulting from (n-1) successive convolutions of a pair of points with itself. Thus, a = b = c etc. The points are equally spaced and their amplitudes are the binomial coefficients for terms in the expansion of $(1+1)^n/2^n$. The diffraction pattern is simply $\cos^n au$. The position of the zeros and of the points where the pattern is unity is the same regardless of n. If the spacing 2a equals or exceeds λ, strong higher orders are formed as in the case of the diffraction grating. When 2a is less than $\lambda/2$ the diffraction pattern is a monotonic decreasing function approximating a Gauss Error curve, but the array is too weakly illuminated near the ends to be practical.

3. *Grating of 2^n Points* (Fig. 15)

Again, if b = 2a, c = 2b, etc. we find that (n-1) convolutions produce a diffraction grating of 2^n points of equal amplitude equally spaced. The resulting pattenn is

$$g(u) = \cos au \cos 2au \cos 4au \text{ etc.} \tag{56}$$

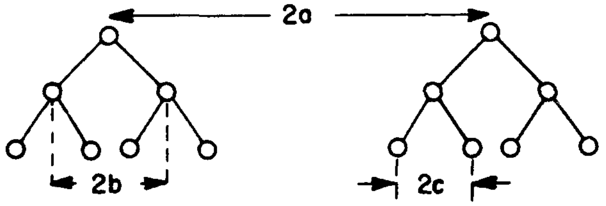
Since a high density of points is equivalent to a uniform line source, it follows that the pattern for a uniform line source of width 2w can be expressed as an infinite continued product.

$$\frac{\sin wu}{u} = \cos\frac{wu}{2} \cdot \cos\frac{wu}{4} \cdot \cos\frac{wu}{8} \cdots$$

4. *Grating of n Equal Points as a Convolution* (Fig. 16)

A simple derivation of the diffraction grating formula[13] is obtained as follows. Let $f_n(x)$ designate an array of n equidistant point

Types of $f(x)$	$g(u)$

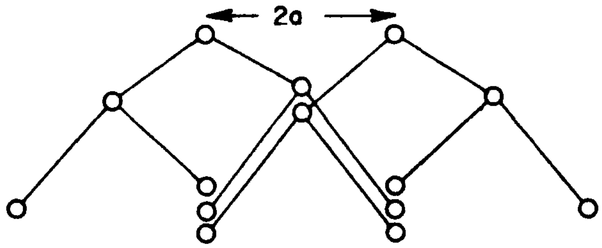


$\cos au$

$\cos au \cos bu$

$\cos au \cos bu \cos cu$

Figure 13. Successive Convolutions of Pairs of Points
Separations 2a, 2b, 2c, etc.

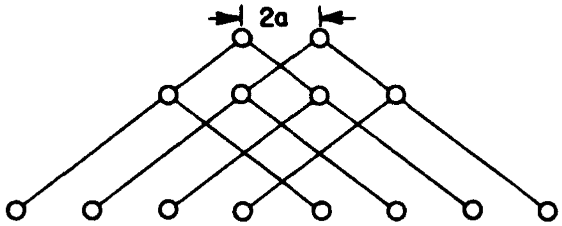


$\cos au$

$\cos^2 au$

$\cos^3 au$

Figure 14. Binomial Distribution
Separations a = b = c, etc.

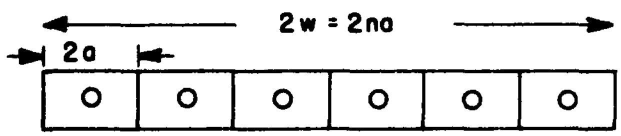


$\cos au$

$\cos au \cos 2au$

$\cos au \cos 2au \cos 4au$

Figure 15. Uniform Grating of 2^n Points
Separations b = 2a, c = 2b, etc.



$$g_n(u) \frac{\sin au}{u} = \frac{\sin wu}{u}$$

$$g_n(u) = \frac{\sin wu}{\sin au} = \frac{\sin nau}{\sin au}$$

Figure 16. Diffraction Grating of n Points

FIGURES 13 - 16. CONVOLUTIONS OF PAIRS OF POINTS sources of constant amplitude separated by intervals 2a. Then let each point be spread into a slit of width 2a. The result is one long slit of width $2w = 2na$, the diffraction pattern of which is $(\sin wu)/u = (\sin nau)/u$. But the long slit was formed from the convolution of the point pattern $f_n(x)$ and the single slit of width 2a, so that the resultant pattern can also be expressed as the product of the point pattern $g_n(u)$ and the single slit pattern $(\sin au)/u$. On equating the two expressions,

$$g_n(u)\,\frac{\sin au}{u} = \frac{\sin nau}{u} \tag{57}$$

Hence

$$g_n(u) = \frac{\sin wu}{\sin au} = \frac{\sin nau}{\sin au} \tag{58}$$

This reduces to n when au is zero or a multiple of π, i.e. when waves from adjacent openings differ by a whole number of waves. The pattern is zero whenever wu is a multiple of π. This causes (n-2) side lobes to appear between each of the strong peaks. (See any optics text for illustrations). Midway between the strong peaks, the envelope of $g_n(u)$, which is $1/(\sin au)$, is unity so that the minimum side lobe has a relative amplitude of $1/n$ or a relative intensity of $(1/n)^2$. The corresponding intensity pattern from the slit of width 2w would have been $(2/\pi)^2$ as much, or 4 db lower.

5. *Grating of n Equal Points as a Product*

The diffraction grating can also be considered as an infinite series of points with spacing 2a, which is observed through a window that is transparent inside the limits $\pm$ w and opaque outside. See Figs. 17, 18. Since f(x) is the product of two functions the diffraction pattern is the convolution of their patterns according to (45)

$$g(u) = \int g_1(\alpha)\, g_2(u-\alpha)\, d\alpha \tag{59}$$

Now the pattern for the infinite grating degenerates to the series of equally spaced points shown in Fig. 18, so that it acts as a window of vertical transparent lines of spacing $u = \pi/a$. Thus, to a first approximation, the pattern is the single slit pattern traced by the line nearest the peak. A. L. Patterson has applied this point of view to the diffraction of x-rays by small crystals.[14]

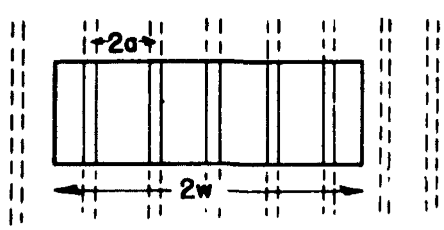


Fig. 17. Grating as Product of Infinite Grating and Window of Width 2w

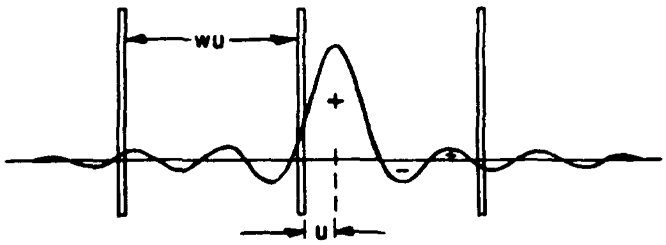


Fig. 18. Grating Pattern as the Convolution of (sin au)/u and the series of equidistant lines of separation $\Delta\varphi = au = \pi$

6. *Grating with Finite Slits or Dipoles*

Since an actual diffraction grating consisting of n slits of width 2b is the convolution of the point grating $f_n(x)$ and the slit of width 2b, its diffraction pattern is simply the product of (58) by the single slit pattern (sin bu/u).

$$g(u) = g_n(u) \frac{\sin bu}{u} = b \frac{\sin nau}{\sin au} \frac{\sin bu}{bu} \tag{60}$$

At the peak this reduces to nb which is the total opening of the grating. Obviously, any other pattern such as that of a dipole can be substituted for the pattern from a single opening. In any case it cannot effect the close side lobes but does help diminish the higher orders.

G. CONVOLUTIONS OF LINE SOURCES IN ONE DIMENSION

In the following examples the convolution usually overlaps. Fig. 19 shows how a uniform line source of width 2a can be split into a number of sections, each of which is spread out into a second uniform line source of width 2b. We will not worry much about the normalization constant but will usually let $g(0) = 1$ which implies that the area of $f(x)$ is normalized to unity.

1. *Trapezoidal Illumination* (Fig. 19)

The result of the convolution of Fig. 19 is a trapezoidal amplitude distribution, the two bases being 2(b+a) and 2(b-a). The diffraction pattern is proportional to the product of the patterns of the two line sources.

$$g(u) = g_1 g_2 = \frac{\sin au}{au} \frac{\sin bu}{bu} \tag{61}$$

Since the numerator cannot exceed unity, the amplitude of the side lobes cannot exceed a fraction $1/(abu^2)$ of the peak.

$$\frac{1}{aubu} = \frac{\lambda^2}{ab} \frac{1}{(2\pi \sin\theta)^2} \tag{62}$$

Types of f(x)	g(u)

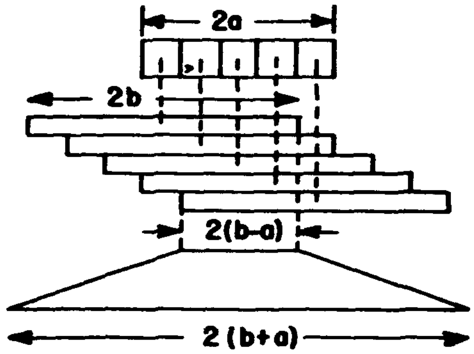


Figure 19. Trapezoid

$$g_1(u) = \frac{\sin au}{au}$$

$$g_2(u) = \frac{\sin bu}{bu}$$

$$g_1 g_2 = \frac{\sin au}{au}\,\frac{\sin bu}{bu}$$

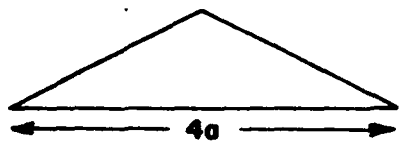


Figure 20. Triangle

$$\left[\frac{\sin au}{au}\right]^2$$

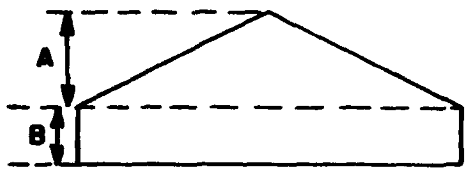


Figure 21. Gable

$$\frac{A}{2}\left[\frac{\sin au}{au}\right]^2 + B\,\frac{\sin 2au}{2au}$$

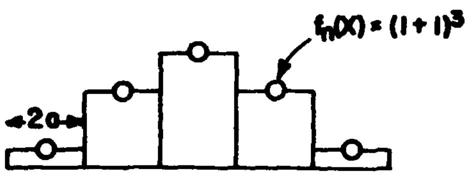


Figure 22. Step Approximation
$f_n(x) = (1+1)^4 = 1,4,6,4,1$

$$g_n(u)\ \frac{\sin au}{au}$$

$$= \cos^4 au\ \frac{\sin au}{au}$$

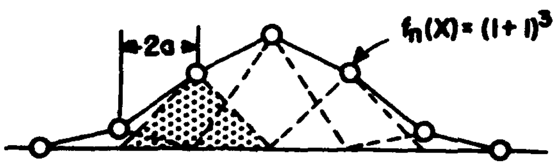


Figure 23. Chord Approximation
$f_n(x) = (1+1)^4$

$$g_n(u)\left[\frac{\sin au}{au}\right]^2$$

$$= \cos^4 au\left[\frac{\sin au}{au}\right]^2$$

FIGURES 19 - 23. CONVOLUTIONS OF UNIFORM LINE SOURCES

Thus, the intensity decreases as $(\sin\theta)^{-4}$, due largely to the taper at the ends. Convolutions of three line sources will decrease at an even greater rate.

2. *Triangular Illumination* (Fig. 20)

When a = b, the trapezoid degenerates to the triangle of Fig. 20 with an overall width of 4a and a pattern

$$g(u) = \left[\frac{\sin au}{au}\right]^2 \tag{63}$$

3. *Gable Illumination* (Fig. 21)

The gable amplitude illumination Fig. 21 is the sum of a triangle of height A and a rectangle of height B. The pattern is:

$$\frac{A}{2}\left(\frac{\sin au}{au}\right)^2 + B\,\frac{\sin 2\,au}{2au} \tag{64}$$

4. *Approximation of f(x)* by its Value at n Equidistant Points*

If an amplitude function f(x) is listed at n equidistant points $f_n(x)$, the diffraction pattern $g_n(u)$ corresponding to $f_n(x)$ will show strong minima and maxima analogous to the orders of a diffraction grating. In order to keep the errors small the phase between adjacent points must be kept small. These errors are partially corrected in the methods described below.

5. *Approximation of f(x) by Steps*

In Fig. 22 the amplitude function f(x) is divided into rectangular steps of equal width 2a and heights f_n equal to the value of the ordinate at the midpoints of the intervals. The resulting approximation is the convolution of the point function $f_n(x)$ and the uniform line source of width 2a, so that the approximate pattern is the product of the transform of $f_n(x)$ and the single slit:

$$g_n(u)\,\frac{\sin au}{au} \tag{65}$$

The $g_n(u)$ is calculated exactly as in the preceding paragraph. The added factor (sin au)/au improves the approximation by suppressing the orders introduced by the grating, $f_n(x)$. Fig. 22 is drawn for the case in which $f_n(x)$ is proportional to 1, 4, 6, 4, 1, the binomial expansion $(1+1)^3$.

6. *Approximation of f(x) by Chords*

Fig. 23 shows how a function, which is zero at its end points, can be approximated by a series of connecting chords. These are equivalent to the convolution of $f_n(x)$ and a triangle of width 4a. The diffraction pattern is the product

$$g_n(u)\left[\frac{\sin au}{au}\right]^2 \tag{66}$$

Should $f(x)$ have end points not zero, say f_1 and f_n, then a straight line function $(A+Bx)$ connecting f_1 and f_n can first be subtracted from $f(x)$. The pattern of $(A+Bx)$ is then calculated independently and added to the solution (66).

H. EFFECT OF A GAP IN A LINE SOURCE

Occasionally an array is constructed as in Fig. 24 with a narrow gap at the center. The effect of the gap is to increase the intensity of the first side lobe from a fraction p^2 to a fraction approximating

$$(p+q)^2 / (1-q)^2 \tag{67}$$

where q is the fraction of the area of the amplitude function $f(x)$ cut out by the gap.

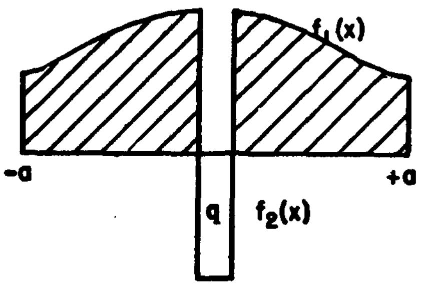


Fig. 24 Gap in Array
Ratio of Areas is q

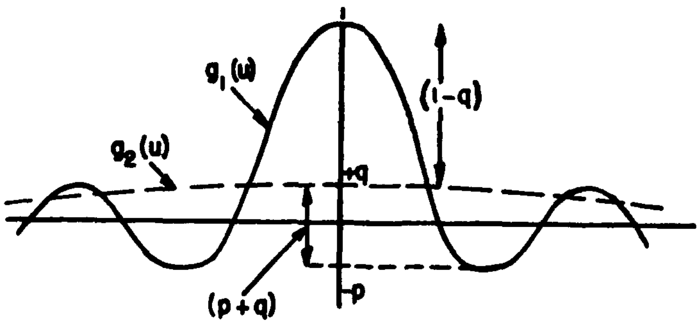


Fig. 25 Gap Increases First Side Lobe by Raising Axis

This is due to the fact that the actual amplitude $f(x)$ of the array is the difference of the original function $f_1(x)$ and the gap $f_2(x)$. The pattern $g_2(u)$ for the gap is so broad that it has the effect of elevating the axis of Fig. 25 by the fraction q. Since the amplitude of the first side lobe was originally -p, the new side lobe has a relative amplitude $-(p+q)/(1-q)$, and a relative intensity of $(p+q)^2/(1-q)^2$. The second side lobe, on the contrary, is reduced and may even vanish.

This formula holds exactly when the central point source is blocked out from an odd-numbered n-point diffraction grating, in which case $q = 1/n$. The intensity of the side lobes midway between two orders was previously $1/n^2$. The odd ones are now increased to $4/(n-1)^2$ while the even ones vanish. When applied to a 3 point array the relative intensity of the one and only side lobe increases from 1/9 to 1.

I. INTEGRATION BY PARTS

This method can be employed when there are discontinuities in $f(x)$ or its derivatives at only a few points, x_k. Between two such points, integration by parts yields:

$$g(u) = \int_{x_1}^{x_2} f(x) e^{iux} dx = \left[e^{iux} \left[\frac{1}{iu} - \frac{D}{(iu)^2} + \frac{D^2}{(iu)^3} \cdots \right] f(x) \right]_{x_1}^{x_2} \qquad (68)$$

On summing over all intervals the contributions cancel except for the negative difference of $D^n f(x)$, $\Delta D^n f(x)$, at each discontinuity.

$$g(u) = \sum_{1}^{k} e^{iux_k} \frac{\Delta}{iu} \left[-1 + \frac{D}{iu} - \left(\frac{D}{iu}\right)^2 + \cdots \right] f(x_k) \qquad (69)$$

We have so far ignored the pattern $\sum A_k e^{iux}$ due to a set of radiant points.

Let

$$F(x) = \int_{0}^{x} f(x)\, dx, \qquad (70)$$

then each radiant point can be considered as a discontinuity in $F(x)$. Since

$$f(x) = DF(x)$$

the complete formula is

$$g(u) = \sum_{1}^{k} e^{iux_k} \Delta \left[1 - \frac{D}{iu} + \left(\frac{D}{iu}\right)^2 - \cdots \right] F(x_k) \qquad (71)$$

where x_k is now the position of either a radiant point or a discontinuity in $f(x)$. Thus, if $D^n F(x)$ is the lowest order derivative containing a discontinuity, the leading term in $g(u)$ will vary as $1/u^n$ and the envelope of the power pattern will vary as $1/u^{2n}$. The corresponding condition on $f(x)$ is that $D^n f(x)$ is the lowest order derivative which becomes infinite. Usually the illumination and its derivatives are continuous except at the ends of the interval. If f(x) is also an even function with a range $\pm$ 1,

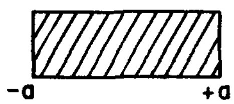
$$g(\varphi) = \frac{\sin\varphi}{\varphi}\left[1 - \left(\frac{D}{\varphi}\right)^2 + \left(\frac{D}{\varphi}\right)^4 - \cdots\right] f(1) + \frac{\cos\varphi}{\varphi}\left[\frac{D}{\varphi} - \left(\frac{D}{\varphi}\right)^3 + \cdots\right] f(1) \quad (72)$$

Several examples of Eqs. (71) and (72) are given in Fig. 26

f(x) g(u)

(1) 

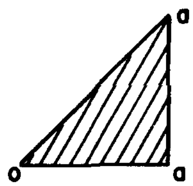
$$e^{iua} + e^{-iau} = 2\cos au$$

(2) 

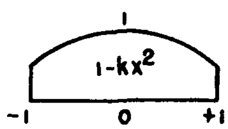
$$\frac{e^{iua} - e^{-iau}}{iu} = \frac{2\sin au}{u}$$

(3) 

$$\frac{e^{iua} - 2 + e^{-iau}}{(iu)^2} = 4\,\frac{(\sin au)^2}{u^2}$$

(4) 

$$\frac{ae^{iau}}{iu} - \frac{(e^{iau} - 1)}{(iu)^2}$$

(5) 

$$\frac{\sin\varphi}{\varphi}\left[(1-k) - \left(\frac{2k}{\varphi}\right)^2\right] + \frac{\cos\varphi}{\varphi}\left[\frac{-2k}{\varphi}\right]$$

Fig. 26 Examples of Diffraction Patterns Derived by Eqs. (71,72)

Examples (1), (2), (3) show discontinuities in F(x) and its first and second derivatives respectively. (See Eq. 71). The results in (2) and (3) could also be obtained from (72). Examples 4 and 5 combine two or more orders of discontinuities.

J. SERIES FOR DIFFRACTION PATTERN[15,16]

For moderately small values of u it is feasible to expand the e^{iux} in the Fourier integral in a power series and then to integrate term by term as follows.

1. *Diffraction Pattern* g(u) *in Terms of Moments of* f(x)

$$g(u) \propto \int f(x)\, e^{iux} dx = \int \left[1 + iux + \frac{(iux)^2}{2!} + \dots \right] f(x)\, dx$$

$$= \int f(x)\, dx + iu \int x f(x)\, dx + \dots + i^n \frac{u^n}{n!} \int x^n f(x)\, dx + \dots \qquad (73)$$

Let us designate by μ_k the kth moment of $f(x)$,

$$\mu_k = \int x^k f(x)\, dx \qquad (74)$$

Then (73) becomes

$$g(u) \propto \sum \frac{i^k \mu_k}{k!} u^k = \mu_o - \frac{\mu_2}{2!} u^2 + \frac{\mu_4}{4!} u^4 - \dots$$

$$+ i \left[\mu_1 u - \frac{\mu_3}{3!} u^3 + \frac{\mu_5}{5!} u^5 - \dots \right] \qquad (75)$$

An even function has only even moments while an odd function has only odd moments. The idea of a moment is common in mechanics. The zero moment μ_o is the area $\int f(x)\, dx$. The center of gravity, c.g., and the square of the radius of gyration, R^2, of a distribution of mass points are also simply expressed in terms of moments.

$$\text{c.g.} = \frac{\int x f(x)\, dx}{\int f(x)\, dx} = \frac{\mu_1}{\mu_o}, \quad R^2 = \frac{\int x^2 f(x)\, dx}{\int f(x)\, dx} = \frac{\mu_2}{\mu_o} \qquad (76)$$

The series (75) converges when $f(x)$ is positive and confined to the range $\pm$ a, for then, $\mu_k \leq a^k \mu_o$, and the terms of the series do not exceed those of the series $\mu_o e^{iau}$, which converges. For the case when $f(x)$ consists of two points of amplitude ½ placed at $\pm$ a, $\mu_k = a^k$ and the series (75) reduces to cos au. Again, for uniform illumination between $\pm$ a

$$\mu_k = \int_{-a}^{a} x^k dx = 2 \frac{a^{k+1}}{k+1} \qquad (77)$$

and

$$g(u) = 2a \left[1 - \frac{a^2 u^2}{3!} + \frac{a^4 u^4}{5!} - \dots \right] = 2a \frac{\sin\varphi}{\varphi}$$

2. *Series for the Power Pattern* $P(u) = g(u)\, \hat{g}(u)^*$

The series for the power diffraction pattern is the sum of the squares of the real and imaginary series of (75) according to (29)

$$P(u) \propto \mu_o^2 - (\mu_o \mu_2 - \mu_1^2) u^2 + \left(\frac{\mu_o \mu_4}{12} - \frac{\mu_1 \mu_3}{3} + \frac{\mu_2^2}{4} \right) u^4 - \qquad (78)$$

$$\mu_0^2 - \mu_0 \mu_2 u^2 + \left(\frac{\mu_0 \mu_4}{12} + \frac{\mu_2^2}{4} \right) u^4$$

*$\hat{g}(u)$ means complex conjugate of $g(u)$

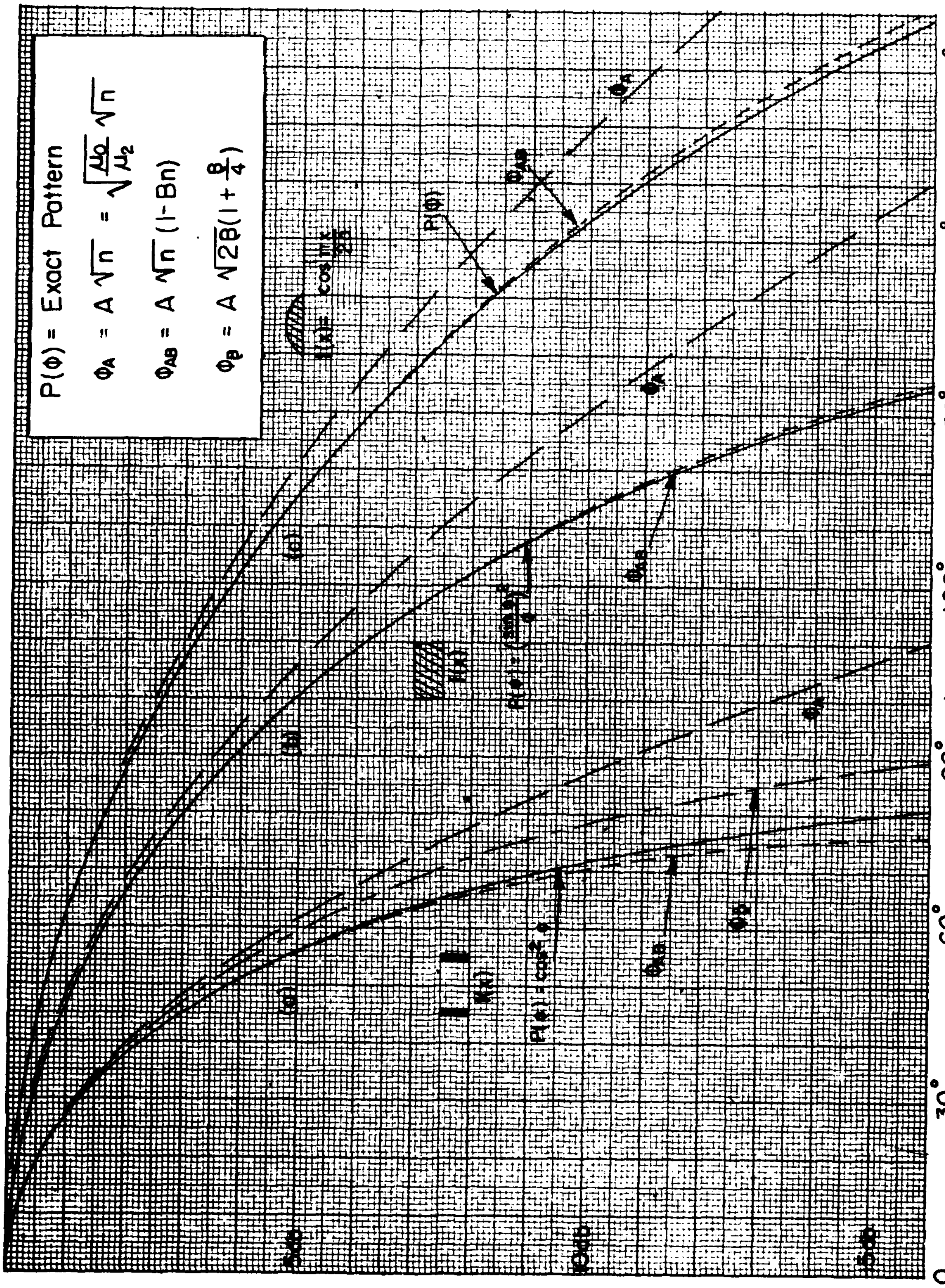


Figure 27. Comparison of Approximations Over the Main Lobe of:
(a) Two Points, (b) Uniform Illumination, (c) Half Cycle Cosine

Let us assume that μ_1 is zero, which is true for an even function, or for any function whose origin is shifted to its center of gravity, (76). On factoring out μ_o^2

$$P(u) \propto 1 - \left(\frac{\mu_2}{\mu_o}\right)u^2 + \frac{1}{4}\left[\left(\frac{\mu_2}{\mu_o}\right)^2 + \frac{\mu_4}{3\mu_o}\right]u^4 - \cdots = 1 - z \tag{79}$$

$-\log P = -\log(1-z) = z$

We may employ the series

$$N = -\log_e P(u) = -\log_e(1-z) = z + \frac{z^2}{2} + \cdots \tag{80}$$

$-\frac{\mu_2}{\mu_o}u^2 - \frac{1}{4}\left[\frac{\mu_2^2}{\mu_o^2} + \frac{\mu_4}{3\mu_o}\right] + \frac{2\mu_2^2}{4\mu_o^2}$

to obtain the power drop N in Napiers.

$$N = \frac{\mu_2}{\mu_o}u^2 + \frac{1}{4}\left(\frac{\mu_2^2}{\mu_o^2} - \frac{\mu_4}{3\mu_o}\right)u^4 = \frac{\mu_2}{\mu_o}u^2\left[1 + \frac{1}{4}\left(1 - \frac{\mu_o\mu_4}{3\mu_2^2}\right)\frac{\mu_2}{\mu_o}u^2\right] \tag{81}$$

The term in parenthesis is a small quantity which approaches zero as P(u) approaches the Gauss Error curve $e^{-(\mu_2/\mu_o)u^2}$. The drop n in db's is 4.343 times N.

$N = au^2 + bu^4$

3. *Widths of Diffraction Patterns*

$u^2 = \frac{N}{a} - \frac{b}{a}\frac{N^2}{a^2}$

The series method can be employed to solve for the widths of diffraction patterns. On reversing [17] the series (81)

$$u^2 = \frac{\mu_o}{\mu_2} N\left[1 - \left(1 - \frac{\mu_o\mu_4}{3\mu_2^2}\right)\frac{N}{4}\right]$$

or

$1.518\sqrt{D}$ $\quad$ $2.303\,D$

$$u = \sqrt{\frac{\mu_o}{\mu_2}}\sqrt{N}\left[1 - \left(1 - \frac{\mu_o\mu_4}{3\mu_2^2}\right)\frac{N}{8}\right]$$

$$= A\sqrt{N}\left[1 - BN\right] \tag{82}$$

where A and B are constants. Eq. 82 is an improvement over that in RL Report T-7[1] which was developed as follows. If β is the fractional drop in amplitude measured from the peak, then g(u) in (75) equals $(1-\beta)$. If we neglect terms above μ_2 and solve for u, then

$$u^* = \sqrt{\frac{\mu_o}{\mu_2}}\sqrt{2\beta} \tag{83}$$

If we include the term in μ_4 but assume that it is the same as for the Gauss Error curve, then the following approximation results

$$u = \sqrt{\frac{\mu_0}{\mu_2}} \quad \sqrt{2\beta} \quad \left(1 + \frac{\beta}{4}\right) \tag{84}$$

This is equivalent to Eqs. (13) and (98) of Report T-7. In the following discussion we use φ = au.

The several approximate formulas are compared in Fig. 27 for the main lobe of the patterns for three illumination functions: (a) two points, (b) uniform illumination and (c) the half cycle cosine. The simplest approximation is φ_A, based on the Gauss Error curve. This is fairly good down to half power. Eq. (84), indicated by φ_β, lies inside φ_A by a fraction which depends only on n. Although φ_β may fit some patterns it cannot compare with the surprisingly accurate approximation φ_{AB} based on (82). The percent error in width using φ_{AB} is shown in Fig. 28. For the case of uniform illumination the error is less than 0.5% down to 25 db. Paradoxically, the reversed series (82) is more accurate than the direct series (81), which in turn is much closer than the original arithmetic series (79).

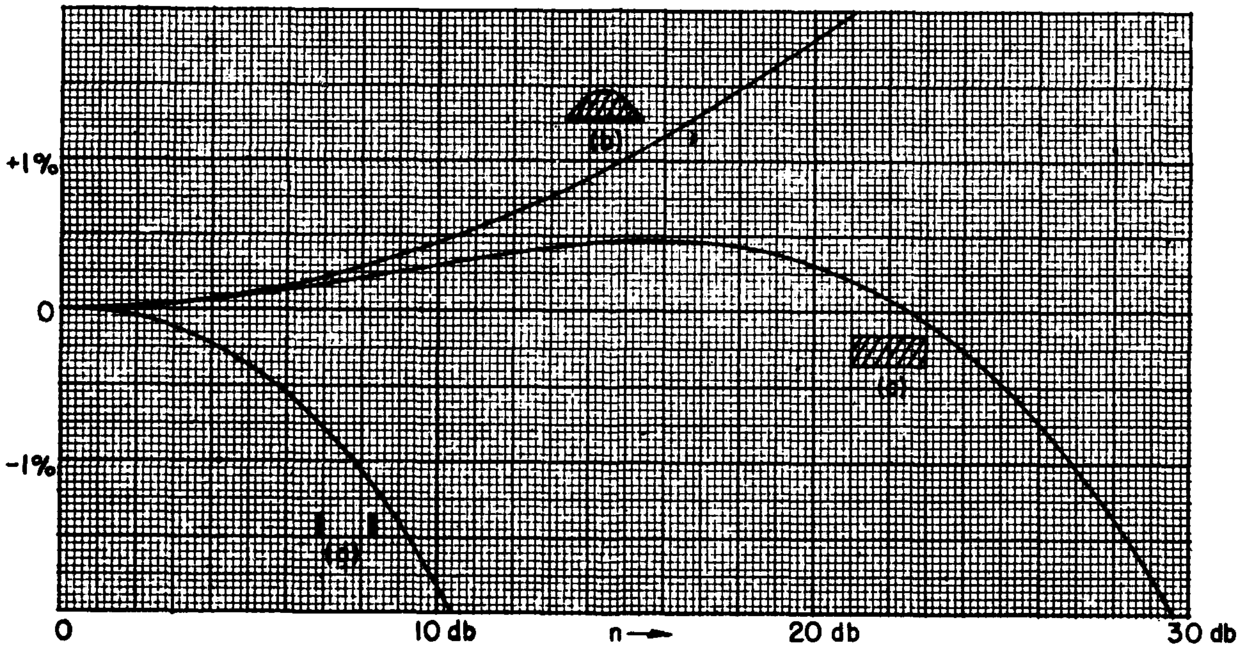


Fig. 28. Percent Error in Width Formula, $W = A\sqrt{n}(1-Bn)$ For Three Types of Illumination

4. *Calculation of Beam Widths*

The beam width W in radians of a diffraction pattern is approximated by

$$W = 2 \sin\varphi \tag{85}$$

which is $(\lambda/\pi)u$. Thus we may write

$$W = A\sqrt{n}(1 - Bn)\frac{\lambda}{d} \tag{86}$$

where λ and the diameter, $d = 2a$, are in like units, and the constants A and B are defined as follows

$$A = 0.305 \quad a\sqrt{\frac{\mu_0}{\mu_2}}, \qquad B = \left[1 - \frac{\mu_0\mu_4}{3\mu_2^2}\right]\frac{1}{34.7} \tag{87}$$

Table 1 Constants For Width Formula

Type of f (x)			 Cos ½ πx
A	.306	.529	.702
B	.0192	.012	.0079
n	6	25	11

Table 1 lists the constants A and B for use in (86). The n in the table is the range in dbs over which the error in width is less than 0.5%.

Each constant is dimensionless. For example, for uniform illumination, $\mu_k = 2a^k/(k+1)$ and

$$W = 0.529\sqrt{n}\left(1 - \frac{n}{86.8}\right)\frac{\lambda}{d} \tag{88}$$

It is an interesting coincidence that when W^o is in degrees, λ in centimeters and d in feet, that $A = 0.995$ or approximately unity Thus

$$W^o\,(\text{uniform illumination}) = \sqrt{n}\left(1 - \frac{n}{87}\right)\frac{\lambda(cm)}{d\,(ft)} \tag{89}$$

5. *Beam Widths of Convolutions*

The pattern of a convolution is the product of two patterns $g_1\,g_2$, which usually approximates a Gauss Error curve even closer than does either pattern separately. We can thus drop the B in (86). Since the second moments are additive, it follows that the width W of the convolution is related to the widths W_1 and W_2 of g_1 and g_2 by the expression

$$\frac{1}{W^2} = \frac{1}{W_1^{\,2}} + \frac{1}{W_2^{\,2}} \tag{90}$$

A simple graphical solution is given in Fig. 29.

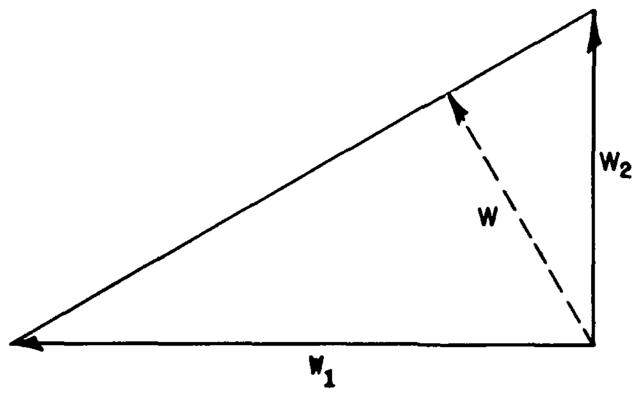


Fig. 29. Relation Between Widths W, W_1, and W_2.

The author wishes to thank Mary Batchelder for the calculations in Figs. 27 and 28, and Pauline Austin for her careful reading of the manuscript.

R. C. Spencer
June 30, 1945

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10 Ref. 1, p 48-50

11 The convolution theorem is usually treated for one dimension only. See Philip Franklin, A Treatise on Advanced Calculus, John Wiley and Sons, 1940, p 498; Titchmarsh, Theory of Fourier Integrals, Oxford, 1937, p 58-60; Raymond E. A. C. Paley and Norbert Wiener, Fourier Transforms in the Complex Domain, Amer. Math. Soc. Colloquium Publications, Vol XIX, p 2. The n-dimensional convolution theorem is discussed by Norbert Wiener under the title Generalized Harmonic Analysis, Acta Mathematica, *24*,(1930) p 117-258, Chap.III.

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